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3 × 3 Lemma and Protomodularity

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The classical 3 × 3 lemma and snake lemma, valid in any abelian category, still hold in any quasi-pointed (the map $0 \to 1$ is a mono), regular, and protomodular category. Some applications are given, in this abstract context, concerning the denormalization of kernel maps and the normalization of internals groupoids (i.e., associated crossed modules). © 2001 Academic Press

Key Words: short exact sequence; short five lemma; 3 × 3 lemma; abelian category; regular and protomodular category.

INTRODUCTION

The notion of exact sequence has an intrinsic meaning in any pointed, regular, and protomodular category [2], among the examples of which there are the category of groups, rings, Lie algebras, Jordan algebras, any variety of Ω-groups, any abelian category of course, the dual of the category of pointed sets, and more generally, any category of internal groups or internal rings in a left exact and regular category, any dual of the category of pointed objects in a topos, see also [10] for the notion of semi-abelian category. Therefore, the question naturally arises whether the classical results concerning the exact sequences still hold in this kind of category, which would put it as a good simple abstract setting for homological algebra in a nonabelian context. We show here that the 3 × 3 lemma and the snake lemma actually hold in it, and even in a slightly larger context, namely, that of quasi-pointed (the map $0 \to 1$ is no more an iso, but only a mono), regular, and protomodular category, defined here as the *sequential* categories. This allows us to integrate as examples any fibre

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$\mathrm{Gr}d_X \mathbb{C}$ of the filtration $(\ )_0$: $\mathrm{Gr}d\,\mathbb{C} \to \mathbb{C}$ associating with each internal groupoid its object of objects, when the basic category $\mathbb{C}$ is left exact and regular. Some applications of these results are given about the "denormalization" of the kernel maps which gives rise to a characterization of some kernel equivalences associated with a morphism and about the "normalization" of the internal groupoids which gives rise to a characterization of the "internal crossed modules" associated with the connected internal groupoids.

For the sake of brevity, we voluntarily restricted to what we could call the "passive" aspect of the 3×3 lemma, meaning by that the situation where all the morphisms are explicitly given. There is clearly some "active" versions of it, when it is possible to create some morphisms from only a part of the 3×3 diagram. This would obviously be the case when the notion of normal monomorphism is clearly conceptually distinguished from that of the kernel map. In this sense, this article is quite complementary to [4], where it is shown precisely that the notion of normal monomorphism also has an intrinsic meaning in any protomodular category, without any right exactness condition (see also [5]). The coordination between the two articles is rightly realized in the context of Barr exact categories [1], i.e., regular categories in which every equivalence relation is effective, that is the kernel equivalence of some map.

The article is organized along the following line:

(1) Quasi-pointed categories.

(2) Protomodular categories.

(3) Sequentiable categories.

(4) The 3×3 lemma.

(5) The snake lemma.

(6) Some applications.

1. QUASI-POINTED CATEGORIES

We call *quasi-pointed* a left exact category $\mathbb{E}$ with an initial object such that the map $0 \to 1$ is a monomorphism.

This implies that, given an object X in $\mathbb{E}$, there is at most one map $X \to 0$ and that the kernel equivalence of this map is the same as the kernel equivalence of the terminal map $X \to 1$, namely, the coarse equivalence $\mathrm{gr}\,X$.

A map will be said to be *trivial* or *null* when it factors (in a unique way) through 0. The kernel of any map $f\colon X \to Y$ is then defined by the following pullback:

$$\begin{tikzcd} K[f] \arrow[r, "\ker f"] \arrow[d] & X \arrow[d, "f"] \\ 0 \arrow[r, "\alpha_Y"'] & Y \end{tikzcd}$$

The cokernel of the map $f\colon X \to Y$ is then any map $q\colon Y \to Q$ which universally trivializes f. This implies that X is above 0, and defines this cokernel as the pushout along f of the map $X \to 0$. When it exists, we shall denote it by $\operatorname{coker} f$ and its codomain by $\mathrm{Coker}[f]$. The cokernel of any map, when it exists, is a regular epimorphism, since in any category with pullbacks: (1) any map $X \to 0$, being split, is a regular epimorphism and (2) the regular epimorphisms (being the quotient of their kernel pairs) are stable by pushouts whenever they exist. It is not the case in any quasi-pointed category that a regular epimorphism is the cokernel of its kernel. Consider the category Sets* of pointed sets, for instance.

EXAMPLE. Suppose $\mathbb{C}$ is a left exact category and denote $\mathrm{Gr}d\,\mathbb{C}$ the category of internal groupoids in $\mathbb{C}$. Let $(\ )_0\colon \mathrm{Gr}d\,\mathbb{C} \to \mathbb{C}$ the functor associating with each groupoid its object of objects. It is a fibration. Clearly $\mathrm{Gr}d_1\,\mathbb{C}$, the fibre above 1, is just $\mathrm{Gp}\,\mathbb{C}$ the category of internal groups in $\mathbb{C}$. Any fibre $\mathrm{Gr}d_X\,\mathbb{C}$ above any object X is thus the category of internal groupoids with fixed objects X. It is quasi-pointed by the kernel equivalence dis X of $1_X\colon X \to X$.

The following result is classical in any quasi-pointed category $\mathbb{E}$:

LEMMA 1. *Given a commutative diagram where the map k is the kernel of the map f:*

$$\begin{tikzcd} K' \arrow[r, "k'"] \arrow[d, "u"'] & X' \arrow[r, "f'"] \arrow[d, "v"] & Y' \arrow[d, "w"] \\ K \arrow[r, "k"'] & X \arrow[r, "f"'] & Y \end{tikzcd}$$

(1) *Suppose w is a monomorphism, then k' is the kernel of f' if and only if the left-hand side square is a pullback.*

(2) *Suppose the right-hand side square is a pullback, then k' is the kernel of f' if and only if u is an isomorphism.*

2. PROTOMODULAR CATEGORIES

We mentioned that in general cokernels and regular epimorphisms do not coincide. This distinction will disappear in the following context.

We denote by $\mathrm{Pt}\,\mathbb{E}$ the category whose objects are the split epimorphisms in $\mathbb{E}$ with a given splitting and morphisms the commutative squares between these data. We denote by $\pi\colon \mathrm{Pt}\,\mathbb{E} \to \mathbb{E}$ the functor associating its codomain with any split epimorphism. As soon as $\mathbb{E}$ has pullbacks, the functor π is a fibration which is called the *fibration of pointed objects*.

The category $\mathbb{E}$ is said to be *protomodular* [2] when π has its change of base functors conservative; i.e., reflecting isomorphisms: when an arrow is mapped onto an isomorphism, it is an isomorphism. When $\mathbb{E}$ is pointed, this condition is equivalent to the split short five lemma, which makes the category Gp of groups the leading example of this notion.

EXAMPLE. When $\mathbb{C}$ is left exact, then any fibre $\mathrm{Grd}_X\,\mathbb{C}$ above an object X shares with the fibre $\mathrm{Grd}_1\,\mathbb{C} = \mathrm{Gp}\,\mathbb{C}$ the property of being protomodular. Any dual of an elementary topos is protomodular.

Remark. In any quasi-pointed protomodular category, a map f is a monomorphism if and only if its kernel is 0. More generally, pullbacks reflect monomorphisms, see [2].

As soon as a functor $F\colon \mathbb{E} \to \mathbb{E}'$ preserves pullbacks and is conservative, $\mathbb{E}'$ protomodular implies $\mathbb{E}$ protomodular. Accordingly, any fibre $\mathrm{Pt}_X\,\mathbb{E}$ of π above an object X is protomodular, as well as any slice category $\mathbb{E}/X$. The protomodularity condition is equivalent to the following one: given a pullback of split epimorphisms, then the pair (u, s') is jointly strongly epic:

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, shift right, "d"'] & U' \arrow[d, shift right, "d'"'] \\ V \arrow[r, "v"'] \arrow[u, shift right, "s"'] & V' \arrow[u, shift right, "s'"'] \end{tikzcd}$$

It follows from that:

PROPOSITION 2. *In a quasi-pointed protomodular category, a map f is a regular epi if and only if it is the cokernel of its kernel.*

Proof. Given a map $f\colon X \to Y$, let us consider the following diagram:

$$\begin{tikzcd} K[f] \times K[f] \arrow[r, "\gamma"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & X \times_Y X \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\ K[f] \arrow[r, "\ker f"'] \arrow[d] & X \arrow[d, "f"'] \\ 0 \arrow[r] & Y \end{tikzcd}$$

The map $K[f] \to 0$, being split, is the quotient of its kernel equivalence. Let s_0 denote the diagonal $X \to X \times_Y X$, the pair (γ, s_0) is then jointly strongly epic. Now, a map g: $X \to Z$ coequalizes the pair (p_0, p_1) if and only if it coequalizes the pairs $(p_0.\gamma, p_1.\gamma)$ and $(p_0.s_0, p_1.s_0)$. In other words, if and only if $g.\ker f$ coequalizes the pair (p_0, p_1), i.e., if and only if $g.\ker f$ factors through 0. Consequently, f is a regular epi if and only if it is the cokernel of its kernel. ∎

Whence, the following definition [2]:

DEFINITION 3. Given a quasi-pointed protomodular category, a short exact sequence is a trivial sequence (i.e., with the composite trivial) such that k is the kernel of f and f is the cokernel of k. We shall picture it in

$$\star \longrightarrow K \xrightarrow{k} X \xrightarrow{f} Y \longrightarrow \star.$$

3. SEQUENTIABLE CATEGORIES

DEFINITION 4. We shall call sequentiable a category which is quasi-pointed, regular, and protomodular.

This means [1] that, moreover, every effective equivalence relation (i.e., kernel equivalence of some map) has a quotient and that regular epimorphisms are stable by pullback.

Remark. In this protomodular context, any kernel map has a cokernel and the classical epi-mono factorization of a map f: $X \to Y$ is obtained in the following way: take its kernel k: $K \to X$ and then take the cokernel q: $X \to Q$ of k. The factorization $Q \to Y$ is a monomorphism.

EXAMPLE. When $\mathbb{C}$ is regular, then any fibre $\mathrm{Gr}d_X\,\mathbb{C}$ above an object X is still regular. Consequently, when $\mathbb{C}$ is left exact and regular, then any fibre $\mathrm{Gr}_X\mathbb{C}$ above an object X is sequentiable.

When $\mathbb{E}$ is regular and protomodular, any fibre $\mathrm{Pt}_X\,\mathbb{E}$ is sequentiable. When $\mathbb{E}$ is sequentiable, any slice category $\mathbb{E}/X$ is sequentiable. The dual of any elementary topos $\mathbb{C}$ being regular and protomodular, the fibres $\mathrm{Pt}_X\,\mathbb{C}^{\mathrm{op}} = (\mathrm{Pt}_X\,\mathbb{C})^{\mathrm{op}}$ are sequentiable.

Let us recall [2] that, in the presence of regularity, protomodularity is equivalent to the following condition we shall need later on:

PROPOSITION 5. *Suppose the category* $\mathbb{E}$ *is left exact and regular, it is protomodular if and only if the following condition holds: given a commuta-*

tive diagram with the middle vertical map a regular epimorphism:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

if the left-hand side and the total rectangle are pullbacks, then the right-hand square is a pullback.

This gives a nice way of checking when a reflexive graph is a kernel equivalence:

COROLLARY 6. *Given an augmented reflexive graph, it is the kernel equivalence of its augmentation* f, *if and only if the map* $d_1.\ker d_0$ *is* $\ker f$.

Proof. Let us consider the following augmented reflexive graph:

$$\begin{tikzcd} Y & X \arrow[l, "f"'] \arrow[r] & G \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] \end{tikzcd}$$

Now, consider the following commutative diagram:

$$\begin{tikzcd} K[d_0] \arrow[r, "\ker d_0"] \arrow[d] & G \arrow[r, "d_1"] \arrow[d, "d_0"'] & X \arrow[d, "f"] \\ 0 \arrow[r] & X \arrow[r, "f"'] & Y \end{tikzcd}$$

The middle vertical map is a regular epimorphism as being split. Then, apply the previous proposition. ∎

More generally, we get the short five lemma in full generality through the following kind of converse of Lemma 1.

PROPOSITION 7. *Given, in any sequentiable category, a commutative diagram where* k *is the kernel of* f *and the upper row exact*:

$$\begin{tikzcd} \bigstar \arrow[r] & K' \arrow[r, "k'"] \arrow[d, "u"'] & X' \arrow[r, "f'"] \arrow[d, "v"] & Y' \arrow[r] \arrow[d, "w"] & \bigstar \\ \bigstar \arrow[r] & K \arrow[r, "k"'] & X \arrow[r, "f"'] & Y & \end{tikzcd}$$

(1) *if the left-hand side square is a pullback, then* w *is a mono.*

(2) *if* u *is an isomorphism, then the right-hand side square is a pullback.*

(3) *if* u *and* w *are isomorphisms, then* v *is an isomorphism (short five lemma).*

Proof. (1) The left-hand side square being a pullback, k' is the kernel of $f.v = w.f'$. The upper row is exact, thus $w.f'$ is the epi-mono factorization of the map $f.v$, and w is a mono.

(2) Now, consider the following commutative diagram:

$$\begin{tikzcd} K' \arrow[r, "k'"] \arrow[d] & X' \arrow[r, "v"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y' \arrow[r, "w"'] & Y \end{tikzcd}$$

The middle vertical map is regular, the left-hand side square is a pullback. But $y.k' = k.u$ and u is an isomorphism. Consequently, the total rectangle is also a pullback. Then the right-hand side square is a pullback.

(3) Furthermore, when w is an iso, the map v is itself an iso. ∎

Our aim now is to prove that the 3×3 lemma holds in full generality in any sequentiable category. For that, we need two preliminary results.

PROPOSITION 8. *Given a morphism of the previous kind*:

$$\begin{tikzcd} \star \arrow[r] & K' \arrow[r, "k'"] \arrow[d, "u"'] & X' \arrow[r, "f'"] \arrow[d, "v"] & Y' \arrow[r] \arrow[d, "w"] & \star \\ \star \arrow[r] & K \arrow[r, "k"'] & X \arrow[r, "f"'] & Y & \end{tikzcd}$$

suppose w is an isomorphism, then u is a regular epimorphism if and only if v is a regular epimorphism.

Proof. The map w being an iso, the left-hand side square, following Lemma 1, is a pullback. So, when v is a regular epi, u is a regular epi.

Conversely, let us denote v': $X' \to Q$ the cokernel of the kernel of v, and v'' is the monomorphism such that $v = v''.v'$. The map u being a regular epi, there is a factorization ϕ: $K \to Q$ such that $\phi.u = v'.k'$ and $v''.\phi = k$. Now, let us set $\psi = f.v''$. Then, $w.f' = \psi.v'$. The map w being an iso and f' being a regular epi, the map ψ is a regular epi. On the other hand, v'' being a mono, the following square is a pullback:

$$\begin{tikzcd} K \arrow[r, "\phi"] \arrow[d, "1_K"'] & Q \arrow[d, "v''"] \\ K \arrow[r, "k"'] & X \end{tikzcd}$$

Thus, according to Lemma 1, the map φ is the kernel of ψ. Now, the short five lemma implies that v'' is an iso and $v = v''.v'$ a regular epi. ∎

More generally, starting from a diagram of the previous kind, we can construct the following diagram ($*$), where the lower right-hand side square is a pullback,

$$\begin{tikzcd}
K' \arrow[r, "k'"] \arrow[d, "u"'] & X' \arrow[r, "f'"] \arrow[d, "v_1"] & Y' \arrow[d, "1_{Y'}"] \\
K \arrow[r, "h"'] \arrow[d, "1_K"'] & Z \arrow[r, "g"'] \arrow[d, "v_2"] & Y' \arrow[d, "w"] \\
K \arrow[r, "k"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

the map g is then a regular epi since f' is so, and the middle row is exact. Moreover, the upper left-hand side square is a pullback.

COROLLARY 9. *Consider any commutative diagram as above*:

$$\begin{tikzcd}
\star \arrow[r] & K' \arrow[r, "k'"] \arrow[d, "u"'] & X' \arrow[r, "f'"] \arrow[d, "v"] & Y' \arrow[r] \arrow[d, "w"] & \star \\
\star \arrow[r] & K \arrow[r, "k"'] & X \arrow[r, "f"'] & Y &
\end{tikzcd}$$

(1) *when w and u are regular epimorphisms, then v is a regular epimorphism.*

(2) *When w is a monomorphism, then u is a mono if and only if v is a mono.*

Proof. Straightforward considering the decomposition ($*$) and the previous results. ∎

COROLLARY 10. (1) *When u is a regular epimorphism, then the restriction $K(f')$: $K[v] \to K[w]$ of the map f' to the kernels is a regular epimorphism.*

(2) *When v and w are split epimorphisms and the right-hand square commutes with the splittings, then the restriction $K(f')$: $K[v] \to K[w]$ of the map f' to the kernels and the extension φ': $X'[v] \to Y'[w]$ of the map f' to the kernel equivalences of v and w are necessarily regular epimorphisms.*

Proof. (1) The map u being a regular epi, such is the map v_1. But the following square is a pullback, where α: $K[w] \to Z$ is the unique map which is the kernel of v_2 and such that $g.\alpha = \ker w$. Consequently, $K(f')$

is a regular epi:

$$
\begin{tikzcd}
K[v] \arrow[r, "K(f')"] \arrow[d, "\ker v"'] & K[w] \arrow[d, "\alpha"] \\
X' \arrow[r, "v_1"'] & Z
\end{tikzcd}
$$

(2) When the right-hand square commutes with the splittings, the map u is split and then a regular epi, consequently the map $K(f')$: $K[v] \to K[w]$ is a regular epi. Now, consider the following diagram:

$$
\begin{tikzcd}
X \arrow[d, "f"'] & X' \arrow[l, "v"'] \arrow[d, "f'"'] \arrow[r, shift left=2, leftarrow, "p_0"] \arrow[r] \arrow[r, shift right=2, leftarrow, "p_1"'] & X'[v] \arrow[d, "\phi'"] & K[p_0] \arrow[l, "\ker p_0"'] \arrow[d, "K(\phi')"] \\
Y \arrow[r, "w"] & Y' \arrow[r, shift left=2, leftarrow, "p_0"] \arrow[r] \arrow[r, shift right=2, leftarrow, "p_1"'] & Y'[w] & K[p_0] \arrow[l, "\ker p_0"]
\end{tikzcd}
$$

then, according to Corollary 6, $K[p_0] = K[v]$ and $K(\varphi') = K(f')$. Now, $K(f')$ and f' are regular epimorphisms, and, according to Corollary 9, the map φ' is a regular epi. ▮

As an immediate consequence, we get that the regular epimorphisms in the category $\mathrm{Gr}\, d\mathbb{E}$ of internal groupoids in a sequentiable category $\mathbb{E}$ are those internal functors whose underlying morphisms of reflexive graphs are (componentwise) regular epimorphisms. A protomodular category being always Mal'cev (see [4]), this could have been indirectly derived from [8], but in a less limpid way.

We shall also need the following technical result:

PROPOSITION 11. *Suppose we have a commutative diagram with k' the kernel of f', and the lower row null*:

$$
\begin{tikzcd}
\star \arrow[r] & K' \arrow[r, "k'"] \arrow[d, "u"'] & X' \arrow[r, "f'"] \arrow[d, "v"] & Y' \arrow[d, "w"] \\
& K \arrow[r, "k"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

The previous decomposition $(*)$ is still possible via the right-hand side lower pullback. Suppose the map v_1 is a regular epimorphism and the upper left-hand side square is a pullback, then the map k is the kernel of f.

Proof. From the fact that the upper left-hand side square is a pullback, we can derive that the map h is a mono since k' is a mono and the category $\mathbb{E}$ is protomodular.

Now, let us take $a\colon A \to X$ the kernel of f. The lower right-hand side square being a pullback, there is a unique map $a'\colon A \to Z$ which is the kernel of g and satisfies $v_2.a' = a$. Consequently, there is a unique map φ: $K \to A$ such that $a'.\varphi = h$, and h being a mono, this φ is itself a mono. According to Lemma 1, the following square is a pullback:

$$
\begin{tikzcd}
K' \arrow[r, "k'"] \arrow[d, "\phi . u"'] & X' \arrow[d, "v_1"'] \\
A \arrow[r, "a'"'] & Z
\end{tikzcd}
$$

and therefore $\varphi.u$ is a regular epi since it is the case for v_1, which implies that φ is itself a regular epi. Thus, φ is an isomorphism. Now, $a.\varphi = v_2.a'.\varphi = v_2.h = k$ and k is the kernel of f. ∎

4. THE 3 × 3 LEMMA

THEOREM 12. *Suppose given the following commutative diagram in a sequentiable category, with the three rows exact*:

$$
\begin{tikzcd}
\star \arrow[r] & K'' \arrow[r, "k''"] \arrow[d, "u'"'] & X'' \arrow[r, "f''"] \arrow[d, "v'"] & Y'' \arrow[r] \arrow[d, "w'"] & \star \\
\star \arrow[r] & K' \arrow[r, "k'"'] \arrow[d, "u"'] & X' \arrow[r, "f'"'] \arrow[d, "v"] & Y' \arrow[r] \arrow[d, "w"] & \star \\
\star \arrow[r] & K \arrow[r, "k"'] & X \arrow[r, "f"'] & Y \arrow[r] & \star
\end{tikzcd}
$$

Suppose the middle column is trivial. If two among the three columns are exact, then the third one is exact.

Proof. The map f'' being a regular epi and the map k being a mono, the nullity of the middle column implies the nullity of the two others. On the other hand, let us introduce the previous decomposition $(*)$. Its middle row is still exact, and its upper left-hand side square is a pullback.

(1) Suppose the two last columns are exact.

The map w' being a mono, the upper left-hand square is a pullback. The map v' being the kernel of v, the map u' is then the kernel of $v.k' = k.u$. But k is a mono, therefore u' is the kernel of u.

On the other hand, u is a regular epi since f'' is a regular epi, according to Corollary 10.

(2) Suppose the two extremal columns are exact.

The last column being exact, the map w is a regular epi. The first column being exact, the map u is a regular epi, and following Corollary 9, such is v. To show that v' is the kernel of v, it is enough to prove that the following square is a pullback,

$$\begin{tikzcd} X'' \arrow[r, "v'"] \arrow[d, "f''"'] & X' \arrow[d, "v_1"] \\ Y'' \arrow[r, "t"'] & Z \end{tikzcd}$$

where the map $t\colon Y'' \to Z$ is the kernel of v_2 and satisfies $g.t = w'$. Now, consider the following diagram $(**)$:

$$\begin{tikzcd} K'' \arrow[r, "k''"] \arrow[d] & X'' \arrow[r, "v'"] \arrow[d, "f''"] & X' \arrow[d, "v_1"] \\ 0 \arrow[r] & Y'' \arrow[r, "t"'] & Z \end{tikzcd}$$

The left-hand square is a pullback since the first row is exact, the middle vertical map f'' is a regular epi, and the total rectangle is equal to the following one:

$$\begin{tikzcd} K'' \arrow[r, "u'"] \arrow[d] & K' \arrow[r, "k'"] \arrow[d, "u"] & X' \arrow[d, "v_1"] \\ 0 \arrow[r] & K \arrow[r, "h"'] & Z \end{tikzcd}$$

But this total rectangle is a pullback, as made of two pullbacks, the left-hand side one meaning that the first column is exact. Consequently, the right-hand square in the diagram $(**)$ is a pullback.

(3) Suppose the two first columns are exact.

Then, v is a regular epi. Thus, $w.f' = f.v$ is a regular epi, and such is w. On the other hand, u is a regular epi and, according to Proposition 8, the map v_1 is a regular epi. Moreover, the following square is a pullback, exactly for the same reasons as previously in 2:

$$\begin{tikzcd} X'' \arrow[r, "v'"] \arrow[d, "f''"'] & X' \arrow[d, "v_1"] \\ Y'' \arrow[r, "t"'] & Z \end{tikzcd}$$

Now applying Proposition 11 to the diagram determined by the two last columns, the map w' is the kernel of w. ▮

Actually, a careful analysis of the previous proof shows that:

PROPOSITION 13. *Let us consider a* 3×3 *diagram of the previous kind with the three rows exact and the middle column null. As soon as the first column is exact,* v' *is the kernel of* v *if and only if* w' *is the kernel of* w.

5. THE SNAKE LEMMA

We shall call *proper* any map u whose monomorphic part of its epi-mono factorization is a kernel map.

PROPOSITION 14. *Suppose given a morphism of exact sequences and that the cokernel of the map u exists,*

$$\begin{tikzcd} \star \arrow[r] & K' \arrow[r, "k'"] \arrow[d, "u"'] & X' \arrow[r, "f'"] \arrow[d, "v"] & Y' \arrow[r] \arrow[d, "w"] & \star \\ \star \arrow[r] & K \arrow[r, "k"'] & X \arrow[r, "f"'] & Y \arrow[r] & \star \end{tikzcd}$$

there is a connecting morphism d: $K[w] \to \mathrm{Coker}[u]$ *such that* $d.K(f')$ *is trivial. Moreover, when u is proper, this trivial sequence is exact at* $K[w]$.

When the cokernel of the map v *exists, the map* $\mathrm{Coker}(k).d$ *is trivial. Moreover, when* v *is proper, then this trivial sequence is exact at* $\mathrm{Coker}[u]$.

When the cokernel of the map w exists, and the maps v *and* w *are proper, then the sequence* $\mathrm{Coker}(f).\mathrm{Coker}(k)$ *is exact at* $\mathrm{Coker}[v]$.

Proof. Let us denote $\ker w$: $K[w] \to Y'$ the kernel of w and let us denote $K(f')$: $K[v] \to K[w]$ the restriction of f' to the kernels. Now, let us consider the following pullback:

$$\begin{tikzcd} H \arrow[r, "\phi"] \arrow[d, "h"'] & K[w] \arrow[d, "\ker w"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

Then, h is the kernel of the map $w.f' = f.v$. Thus, there is a map ν: $H \to K$ such that the following square is a pullback:

$$\begin{tikzcd} H \arrow[r, "h"] \arrow[d, "\nu"'] & X' \arrow[d, "v"] \\ K \arrow[r, "k"'] & X \end{tikzcd}$$

On the other hand, there is a unique map κ: $K' \to H$ which is the kernel of the regular epimorphism φ and such that $h.\kappa = k'$. The map φ is then the cokernel of κ. Moreover, $\nu.\kappa = u$. Thus, if coker u denotes the cokernel of u, then coker $u.\nu.\kappa =$ coker $u.u = 0$. Whence a unique map d: $K[w] \to \mathrm{Coker}[u]$ such that $d.\varphi =$ coker $u.\nu$. Now, because of the second mentioned pullback, there is a map γ: $K[v] \to H$ which is the kernel of ν and such that $h.\gamma =$ ker v. It follows from that: $\varphi.\gamma = K(f')$. Therefore, $d.K(f') = d.\varphi.\gamma =$ coker $u.\nu.\gamma = 0$.

Now, let $u = u''.u'$ be the epi-mono factorization of u. If u is proper, apply Corollary 10 to the following morphism of exact sequences,

$$\begin{tikzcd} \star \arrow[r] & K' \arrow[r, "\kappa"] \arrow[d, "u'"'] & H \arrow[r, "\phi"] \arrow[d, "\nu"] & K[w] \arrow[r] \arrow[d, "d"] & \star \\ \star \arrow[r] & Q \arrow[r, "u''"'] & X \arrow[r, "\mathrm{coker}\, u"'] & \mathrm{Coker}[u] \arrow[r] & \star \end{tikzcd}$$

and the factorization $K[v] = \mathrm{Ker}[\nu] \to K[d]$ is a regular epimorphism. Thus, the sequence $d.K(f')$ is exact at $K[w]$.

Suppose now coker v does exist and denote $\mathrm{Coker}(k)$ the extension of k to the cokernels. Then, $\mathrm{Coker}(k).d$ is trivial since:

$\mathrm{Coker}(k).d.\varphi = \mathrm{Coker}(k).$coker $u.\nu =$ coker $v.k.\nu =$ coker $v.v.h = 0$.

Let $v = v''.v'$ be the epi-mono factorization of the map v and let us suppose v proper. Then, v'' is the kernel of coker v. Let ε be the kernel of $\mathrm{Coker}(k)$ and consider the following square:

$$\begin{tikzcd} K \arrow[r, "\mathrm{coker}\, u"] \arrow[d, "k"'] & \mathrm{Coker}[u] \arrow[d, "\mathrm{Coker}(k)"] \\ X \arrow[r, "\mathrm{coker}\, v"'] & \mathrm{Coker}[v] \end{tikzcd}$$

If η is the pullback of v'' along k, then it is also the pullback of ε along coker u. Now, consider the following diagram where τ is such that $\varepsilon.\tau = d$.

Then,

$$\begin{tikzcd}
X' \arrow[d, "v'"'] & H \arrow[l, "h"'] \arrow[r, "\phi"] \arrow[d, "\theta"'] & K[w] \arrow[d, "\tau"] \\
V \arrow[d, "v''"'] & S \arrow[l] \arrow[r, "j"'] \arrow[d, "\eta"'] & T \arrow[d, "\varepsilon"] \\
X & K \arrow[l, "k"] \arrow[r, "\mathrm{coker}\, u"'] & \mathrm{Coker}[u]
\end{tikzcd}$$

The map j is a regular epi since the lower right-hand side square is a pullback. The map θ such that $\eta.\theta = \nu$ is a pullback of v' since the left-hand side rectangle is a pullback, and thus it is a regular epi. Consequently, the map τ is a regular epi and the trivial sequence $\mathrm{Coker}(k).d$ is exact at $\mathrm{Coker}[u]$.

Let w be proper and let $w = w''.w'$ be its epi-mono factorization. Then, w'' is the kernel of coker w and the factorization ψ such that $w''.\psi = f.v''$ is a regular epi. Let $\zeta\colon J \to \mathrm{Coker}[v]$ denote the kernel of $\mathrm{Coker}(f)$, then thanks to Corollary 10, the factorization $\pi\colon K \to J$ is a regular epi since it is the case for the map ψ which satisfies $\psi.v' = w'.f'$. Now, if μ denotes the factorization of $\mathrm{Coker}(k)$ through ζ, it is a regular epi since $\mu.\mathrm{coker}\, u = \pi$. ∎

6. SOME APPLICATIONS

The kernel of a map is classically considered as the *normalization* of its kernel equivalence. We are somehow now going to denormalize some of our previous results on kernel maps and, conversely, to normalize some aspects of internal groupoids.

PROPOSITION 15. *Let us consider the following diagram of augmented reflexive graphs, where the upper graph is the kernel equivalence of f', and the vertical maps are regular epimorphisms*:

$$\begin{tikzcd}
Y' \arrow[d, "w"'] & X' \arrow[l, "f'"'] \arrow[r] \arrow[d, "v"] & X'[f'] \arrow[l, shift right=2, "p_0"'] \arrow[l, shift left=2, "p_1"] \arrow[d, "u"] \\
Y & X \arrow[l, "f"'] \arrow[r] & G \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"]
\end{tikzcd}$$

(1) *The lower graph is the kernel equivalence of f when the kernel extension of this diagram is exact, i.e., produces a kernel pair with its cokernel*:

$$\begin{tikzcd} K[w] & K[v] \arrow[l, "K(f')"'] \arrow[r] & K[u] \arrow[l, shift right=2, "K(p_0)"'] \arrow[l, shift left=2, "K(p_1)"] \end{tikzcd}$$

(2) *When furthermore f' is a regular epi, then the lower graph is the kernel equivalence of f if and only if the kernel extension of this diagram is exact.*

(3) *When the previous diagram determines a morphism of split augmented reflexive graphs, then the lower graph is necessarily the kernel pair of f.*

Proof. (1) Now consider the following diagram:

$$\begin{tikzcd} Y' \arrow[d, "w"'] & X' \arrow[l, "f'"'] \arrow[r] \arrow[d, "v"'] & X'[f'] \arrow[l, shift right=2, "p_0"'] \arrow[l, shift left=2, "p_1"] \arrow[d, "u"] & K[p_0] \arrow[l, "\ker p_0"'] \arrow[d, "K(u)"] \\ Y & X \arrow[l, "f"'] \arrow[r] & G \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] & K[d_0] \arrow[l, "\ker d_0"] \end{tikzcd}$$

then, according to the 3×3 lemma, the following sequence is exact:

$$\star \longrightarrow K[K(p_0)] \xrightarrow{K(\ker u)} K[p_0] \xrightarrow{K(u)} K[d_0] \longrightarrow \star.$$

Then, consider the following one:

$$\begin{tikzcd} \star \arrow[r] & K[K(p_0)] \arrow[r, "K(\ker u)"] \arrow[d, "K(p_1.\ker p_0)"'] & K[p_0] \arrow[r, "K(u)"] \arrow[d, "p_1.\ker p_0"] & K[d_0] \arrow[r] \arrow[d, "d_1.\ker d_0"] & \star \\ \star \arrow[r] & K[v] \arrow[r, "\ker v"'] \arrow[d, "K(f')"'] & X' \arrow[r, "v"'] \arrow[d, "f'"] & X \arrow[r] \arrow[d, "f"] & \star \\ \star \arrow[r] & K[w] \arrow[r, "\ker w"'] & Y \arrow[r, "w"'] & Y \arrow[r] & \star \end{tikzcd}$$

Suppose the kernel extension is exact, then the first column is exact and, according to Proposition 13, the last column is left exact since the central one is so. Therefore, the lower graph is the kernel equivalence of f, according to Corollary 6.

(2) If f' is a regular epi, then the central column is exact, and therefore the third one is exact if and only if the first one is exact. This same Corollary 6 now ends the proof.

(3) There is one circumstance where the previous conditions 1 and 2 are automatically fulfilled, it is when the augmented reflexive graphs are split; i.e., when there are extra maps $s\colon Y \to X$ and $s_1\colon X \to G$ such that $s.f = d_0.s_1$, $s_0.s = s_1.s$, and $d_1.s_1 = 1_X$, and the same kind of maps at the level of f'.

Indeed, the kernel extension of our diagram produces a split augmented reflexive graph. But clearly its reflexive graph part is jointly monic, i.e., is a reflexive relation. Now, a reflexive relation in a protomodular category is always an equivalence relation since a protomodular category is Mal'cev [4]. This equivalence relation is, as such, an internal groupoid in $\mathbb{E}$ and the splitting of the underlying graph produces a section of the factorization map: $[K(p_0), K(p_1)]\colon K[u] \to K[v] \times_{K[W]} K[v]$ (see [3, Proposition 7]) which

is itself a mono. Consequently, this factorization is an isomorphism and our kernel extension is exact. ∎

Conversely, we are now going to investigate the normalization of the internal groupoids.

It is well known that, in the category Gp of abstract groups, the notion of the internal groupoid is equivalent to the notion of the crossed module, see for instance [6], where a crossed module is given by a homomorphism $h\colon H \to X$ and a left action of the group X on the group H such that h is a homomorphism of left actions (X being endowed with the action on itself by conjugation) which moreover satisfies: $h(y) \star z = y.z.y^{-1}$. The homomorphism underlying the crossed module associated with an internal groupoid is actually given in the category Gp by the following very general construction which makes sense, in any quasi-pointed left exact, category $\mathbb{E}$.

Given an internal groupoid X_1, let us consider the following diagram,

$$\begin{tikzcd}
K[d_0] \times K[d_0] \arrow[r, "k"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & X_1[d_0] \arrow[r, "d_2"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & X_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
K[d_0] \arrow[r, "\ker d_0"'] \arrow[d] & X \arrow[r, "d_1"'] \arrow[d, "d_0"'] & X_0 \\
0 \arrow[r] & X_0 &
\end{tikzcd}$$

where the map d_2, here, represents the operation which, in the set theoretical context, would be defined for every pair of arrows (φ, ψ) with same domain by the formula $d_2(\varphi, \psi) = \psi.\varphi^{-1}$.

DEFINITION 16. We shall refer to the map $h = d_1.\ker d_0$ as the normalization of the internal groupoid X_1.

When $\mathbb{E} = \mathrm{Gp}$, the homomorphism underlying the crossed module associated with X_1 is this normalization.

Now, in any case, all the commutative squares of this diagram are pullbacks. As a consequence, the map $s_0.\ker h$: $K[h] \to K[d_0] \times K[d_0]$ is the kernel of $d_2.k$, which will allow us to characterize the normalization of the connected internal groupoids.

PROPOSITION 17. *Given a sequentiable category* $\mathbb{E}$, *then a regular epimorphism* h: $H \to X$ *is the normalization of a groupoid if and only if the map* $s_0.\ker h$: $K[h] \to H \times H$ *is a kernel map*. *When moreover* $\mathbb{E}$ *is pointed*, *these kinds of regular epimorphisms characterize the connected groupoids*.

Proof. A connected groupoid is such that the map $[d_0, d_1]$: $X_1 \to X_0 \times X_0$ is a regular epimorphism. When the basic category is pointed, the normalization of a groupoid X_1 can be given by the following pullback as well:

$$\begin{tikzcd} H \arrow[r] \arrow[d, "h"'] & X_1 \arrow[d, "{[d_0, d_1]}"] \\ X_0 \arrow[r, "s_1"'] & X_0 \times X_0 \end{tikzcd}$$

Consequently, when the groupoid is connected, then its normalization is a regular epi.

Conversely, let h be a regular epimorphism such that the map $s_0.\ker h$: $K[h] \to H \times H$ is a kernel map. Then, consider X_1 the codomain of the cokernel q of $s_0.\ker h$. This produces a reflexive graph $[d_0, d_1]$: $X_1 \to X \times X$ on X such that the commutative squares in the following diagrams are pullbacks, according to Proposition 7:

$$\begin{tikzcd} H \times H \arrow[r, "q"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & X_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ H \arrow[r, "h"'] & X \end{tikzcd}$$

On the one hand, this graph is underlying a groupoid. Indeed the factorization $\bar{q}$: $H \times H \times H \to X_1[d_0]$ is a regular epimorphism since it is produced by the extension of the previous pullback with the first projections to the kernel equivalences of p_0 and d_0. Consequently, the groupoid X_1 appears to be the cokernel in the category $\mathrm{Grd}\,\mathbb{E}$ of the internal functor: $\operatorname{dis} K[h] \to \operatorname{gr} H$, where $\operatorname{dis} Z$ is the kernel equivalence of 1_Z: $Z \to Z$ and $\operatorname{gr} Z$ is the kernel equivalence of the terminal map $Z \to 1$. Moreover, $[d_0, d_1].q = h \times h$ which is a regular epi, therefore $[d_0, d_1]$ is a regular epi and the groupoid X_1 is connected.

On the other hand, the kernel of d_0 being $q.s_1$ we have: $d_1.\ker d_0 = d_1.q.s_1 = h.p_1.s_1 = h$. ∎

It is certainly worth noticing that, when $\mathbb{E} = \mathrm{Gp}$, the previous regular epimorphisms are precisely those which have central kernels, i.e., those which correspond to central extensions [12].

In the wider context of commutative algebra, a number of examples of such internal crossed modules has been studied in [13]. See also [11, 7, 9].

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